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REMARKS ON THE KORTEWEG–DE VRIES EQUATION

BY

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ABSTRACT

We show for the Korteweg–de Vries equation an existence uniqueness theorem in Sobolev spaces of arbitrary fractional order $s \geqq 2$, provided the initial data is given in the same space.

Introduction

Our aim is to present a remark on the existence and uniqueness of solutions of two initial value problems associated with the Korteweg–de Vries (K. d. V.) equation (see for instance [8]): the Cauchy problem and the initial value problem with periodic boundary conditions, i.e., (0.1), (0.2) or (0.2′), (0.3):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \boldsymbol{R},\ t > 0 \quad (\alpha \neq 0), \tag{0.1}$$

$$\frac{\partial^k u}{\partial x^k}(x,t) \to 0,\ |x| \to +\infty,\ k = 0, \cdots, \tag{0.2}$$

$$\frac{\partial^k u}{\partial x^k}(x+1,t) = \frac{\partial^k u}{\partial x^k}(x,t),\ k = 0, \cdots, \tag{0.2'}$$

$$u(x,0) = u_0(x). \tag{0.3}$$

The existence of solutions of these problems in Sobolev spaces of order 1 or 2 was established in [13] using a technique of parabolic regularization, the equation (0.1) being approximated by

$$\frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon\frac{\partial u_\varepsilon}{\partial x} + \alpha\frac{\partial^3 u_\varepsilon}{\partial x^3} + \varepsilon\frac{\partial^4 u_\varepsilon}{\partial x^4} = 0 \qquad (\varepsilon > 0). \tag{0.4}$$

The existence of solutions in all Sobolev spaces of integer order was established in [15], [5] (using again the parabolic regularization) and in [4] using a different technique.

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Our purpose here is to complete these results by showing the existence of solutions in all Sobolev spaces of order $s \geqq 2$, where *s is not necessarily an integer.* A non-optimal result of this type was previously established by the first author in [9] and while this paper was completed a result similar to ours was announced to the authors by J. Bona who uses a technique completely different from ours ([3]).

Section 1 develops a technical inequality. Section 2.1 contains the main result and Section 2.2 gives a few remarks.

1. An inequality

Let $H^s(\mathbf{R})$ denote the real Sobolev space of order s ($s \in \mathbf{R}, s \geqq 0$), defined for instance by Fourier transform

$$\{u \in L^2(\mathbf{R}),\ |\xi|^{s/2}\hat{u} \in L^2\}, \tag{1.1}$$

and which is a Hilbert space for the norm

$$\|u\|_s = \left\{\int_{-\infty}^{+\infty} (1+|\xi|^2)^s\,|\hat{u}(\xi)|^2\,d\xi\right\}^{1/2}. \tag{1.2}$$

Similarly, let $H^s(C)$ denote the real Sobolev space of order s ($s \in \mathbf{R}, s \geqq 0$) on the unit length circle C; among many other definitions, $H^s(C)$ may be characterized as the space of real periodic functions

$$u(x) = \sum_{k=-\infty}^{+\infty} u_k \exp(2i\pi kx), \tag{1.3}$$

such that

$$\left\{\sum_{k=-\infty}^{+\infty} (1+k^2)^s\,|u_k|^2\right\}^{1/2} < \infty. \tag{1.4}$$

The left-hand side of (1.4) is a Hilbert norm for $H^s(C)$, denoted also $\|u\|_s$.

When no ambiguity is possible, and in order to treat simultaneously the two initial value problems mentioned, H^s will denote either $H^s(\mathbf{R})$ or $H^s(C)$.

The Fourier transform of u will be written $\hat{u}$ or $\mathfrak{F}u$. Let $D = d/dx$ and let D^s represent the fractional derivative of order s:

$$D^s u = \mathfrak{F}^{-1}(|\xi|^s\mathfrak{F}u), \quad \text{if } u \in H^s(\mathbf{R}), \tag{1.5}$$

$$D^s u = \sum_{k=-\infty}^{+\infty} u_k\,|k|^s \exp(2i\pi kx) \text{ if } u \in H^s(C),\ u \text{ of type (1.3)}. \tag{1.6}$$

The next lemma will be useful.

LEMMA 1.1. *Let u, v belong to $H^s(\mathbf{R})$ or $H^s(C)$, $s \in \mathbf{R}$, $s > 1$, $\gamma \in \mathbf{R}$, $\gamma > 1/2$. Then*

$$\| D^s(uv) - uD^s v \|_0 \leqq c(\gamma, s)\{\| u \|_s \| v \|_\gamma + \| u \|_{\gamma+1} \| v \|_{s-1}\}. \tag{1.7}$$

REMARK 1.1.

i) The inequality (1.7) is valid in higher dimensions, $u, v \in H^s(\mathbf{R}^n)$ or $H^s(C^n)$, provided $\gamma > n/2$ (same proof).

ii) Inequality (1.7) is rather easy using Leibnitz' formula, when s is an integer. Similar "fractional type Leibnitz formulas" are extensively used in [1].

PROOF OF LEMMA 1.1.

1/ We start with the case $u, v \in H^s(\mathbf{R})$. We have

$$\widehat{D^s(uv)}(\xi) = |\xi|^s \widehat{uv}(\xi)$$
$$= \int_{-\infty}^{+\infty} |\xi|^s \hat{u}(\xi - \xi')\hat{v}(\xi')\, d\xi',$$

and

$$\widehat{uD^s v}(\xi) = \int_{-\infty}^{+\infty} \hat{u}(\xi - \xi')\, |\xi'|^s \hat{v}(\xi')\, d\xi'.$$

The left-hand side of (1.7) is the L^2 norm of $Y = D^s(uv) - uD^s v$, or using Parseval's formula the L^2 norm of $\hat{Y}$,

$$\hat{Y}(\xi) = \int_{-\infty}^{+\infty} (|\xi|^s - |\xi'|^s)\hat{u}(\xi - \xi')\hat{v}(\xi')d\xi'.$$

It is easy to see that

$$\Big|\, |\xi|^s - |\xi'|^s \Big| \leqq c(s)(|\xi - \xi'|^s + |\xi'|^{s-1}|\xi - \xi'|)$$

where $c(s)$ is a constant depending only on s; whence

$$|\hat{Y}(\xi)| \leqq c(s)(Y_1(\xi) + Y_2(\xi))$$

$$\| \hat{Y} \|_0 \leqq c(s)(\| Y_1 \|_0 + \| Y_2 \|_0) \tag{1.8}$$

where

$$Y_1(\xi) = \int_{-\infty}^{+\infty} |\xi - \xi'|^s |\hat{u}(\xi - \xi')|\, |\hat{v}(\xi')|\, d\xi',$$

$$Y_2(\xi) = \int_{-\infty}^{+\infty} |\xi - \xi'|\, |\xi'|^{s-1} |\hat{u}(\xi - \xi')|\, |\hat{v}(\xi')|\, d\xi'.$$

Both functions Y_1, Y_2, are convolution products, and using the convolution inequalities we find

$$\| Y_1 \|_0 \leqq \| \, | \xi |^s | \hat{u}(\xi) | \, \|_0 \| \hat{v} \|_{L^1(\mathbf{R})},$$

$$\| Y_1 \|_0 \leqq \| u \|_s \| \hat{v} \|_{L^1(\mathbf{R})}, \tag{1.9}$$

$$\| Y_2 \|_0 \leqq \| \, | \xi |^{s-1} | \hat{v}(\xi) | \, \|_0 \| \, | \xi | \, | \hat{u}(\xi) | \, \|_{L^1(\mathbf{R}_\xi)},$$

$$\| Y_2 \|_0 \leqq \| v \|_{s-1} \| \widehat{\operatorname{grad} u} \|_{L^1(\mathbf{R})}. \tag{1.10}$$

It remains to estimate the norm $\| \hat{w} \|_{L^1(\mathbf{R})}$ in terms of a Sobolev norm of w. One has

$$\int_{-\infty}^{+\infty} | \hat{w}(\xi) | \, d\xi = \int_{\mathbf{R}} | \hat{w}(\xi) | (1 + | \xi |^2)^{\gamma/2} \frac{d\xi}{(1 + \xi^2)^{\gamma/2}}$$

and by Schwarz' inequality:

$$\int_{-\infty}^{+\infty} | \hat{w}(\xi) | \, d\xi \leqq c'(\gamma) \| w \|_\gamma \tag{1.11}$$

where

$$c'(\gamma) = \left(\int_{-\infty}^{+\infty} \frac{d\xi}{(1 + \xi^2)^\gamma} \right)^{1/2} < +\infty, \quad \text{if } \gamma > 1/2.$$

The proof is now completed, using (1.8)–(1.11) and recalling that $\| \hat{Y} \|_0$ is equal to the left-hand side of (1.7).

2/ In the case $u, v \in H^s(C)$, the proof is a discrete version of the preceding one.

We have

$$D^s(uv) - uD^s v = \sum_{m=-\infty}^{+\infty} \left(\sum_{k+l=m} (| m |^s - | l |^s) u_k v_l) \exp(2i\pi m x) \right).$$

The left-hand side of (1.7) is equal to Z, with

$$Z^2 = \sum_{m=-\infty}^{+\infty} \left(\left| \sum_{k+l=m} (| m |^s - | l |^s) u_k v_l) \right|^2 \right).$$

Since

$$| \, | k + l |^s - | l |^s \, | \leqq c(s) (| l |^{s-1} | k | + | k |^s)$$

we majorize Z by $c(s)(Z_1 + Z_2)$, with

$$Z_1^2 = \sum_{m=-\infty}^{+\infty} \left(\left| \sum_{k+l=m} | k |^s | u_k | | v_l | \right|^2 \right)$$

$$Z_2^2 = \sum_{m=-\infty}^{+\infty} \left(\left| \sum_{k+l=m} | k | | l |^{s-1} | u_k | | v_l | \right|^2 \right)$$

Now, the discrete convolution inequality gives us

$$Z_1 \leqq \left(\sum_{k=-\infty}^{+\infty} | k |^{2s} | u_k |^2 \right) \left(\sum_{l=-\infty}^{+\infty} | v_l | \right)^2$$

$$Z_2 \leqq \left(\sum_{k=-\infty}^{+\infty} | k | | u_k | \right)^2 \left(\sum_{l=-\infty}^{+\infty} | l |^{2(s-1)} | v_l |^2 \right)$$

and we complete the proof as before, observing that

$$\sum_{k=-\infty}^{+\infty} | w_k | \leqq \left(\sum_{k=-\infty}^{+\infty} (1+k^2)^{-\gamma} \right)^{1/2} \left(\sum_{k=-\infty}^{+\infty} (1+k^2)^{\gamma} | w_k |^2 \right)^{1/2}$$

$$\sum_{k=-\infty}^{+\infty} | w_k | \leqq c'(\gamma) \| w \|_\gamma$$

and $c'(\gamma) < +\infty$ if $\gamma > 1/2$.

2. Existence results

2.1. *The main result.*

THEOREM 2.1. *Let s be any real number $\geqq 2$ and let u_0 be given in $H^s(\boldsymbol{R}$ (resp. $H^s(C)$). The initial value problem* (0.1), (0.2) (*resp.* (0.2′)), (0.3) *possesses unique solution u, $u \in L^\infty(0, T; H^s(\boldsymbol{R}))$ (resp. $L^\infty(0, T; H^s(C))$) for any $T > ($ Moreover u is weakly continuous from $[0, \infty)$ with values in $H^s(\boldsymbol{R})$ (resp. $H^s(C)$).*

PROOF. The uniqueness which holds for any $s > 3/2$ is simple and wa established in [12]. The weak continuity follows from $u \in L^\infty(0, T; H^s)$ and result of W. A. Strauss [10]. For the existence we use the parabolic regularizatio as in [12][†]: we consider a parameter, $\varepsilon > 0$, and a sequence $u_{0\varepsilon} \in H^s \cap \mathscr{C}^\infty(\boldsymbol{R}$ with

$$u_{0\varepsilon} \to u_0 \text{ in } H^s, \text{ as } \varepsilon \searrow 0.$$

For each $\varepsilon > 0$ we consider the initial value problem

[†] The results of [13] which were established for the space periodic problem (0.1), (0.2′), (0. extend without modification to the Cauchy problem (0.1), (0.2), (0.3). For compactness argument we use the fact that the injection of $H^1(-M, +M)$ into $L^2(-M, +M)$ is compact, $\forall M$.

$$\frac{\partial u_\varepsilon}{\partial t}+u_\varepsilon\frac{\partial u_\varepsilon}{\partial x}+\alpha\frac{\partial^3 u_\varepsilon}{\partial x^3}+\varepsilon\frac{\partial^4 u_\varepsilon}{\partial x^4}=0 \quad x\in \mathbf{R},\ t>0, \tag{2.1}$$

$$u_\varepsilon(x,0)=u_{0\varepsilon}(x), \tag{2.2}$$

completed with the boundary condition (0.2) or (0.2').

It is well known that this problem possesses a unique smooth solution and it was shown in [13] that

$$u_\varepsilon \text{ remains bounded in } L^\infty(0,T;H^2), \text{ as } \varepsilon \searrow 0. \tag{2.3}$$

Moreover u_ε converges, as $\varepsilon \searrow 0$, to the solution of the K.d.V. initial value problem (0.1), (0.2) (or (0.2')), (0.3).

Now, it remains to show that

$$u_\varepsilon \text{ remains bounded in } L^\infty(0,T;H^s), \text{ as } \varepsilon \searrow 0. \tag{2.4}$$

For that purpose, we apply the operator D^s on both sides of (2.1) and we take the L^2 scalar product with $D^s u_\varepsilon$‡. Observing that $D=\partial/\partial x$ and D^s commute, we obtain

$$\frac{1}{2}\frac{d}{dt}\| D^s u_\varepsilon \|_0^2+\varepsilon \| D^2 D^s u_\varepsilon\|_0^2+(D^s(u_\varepsilon D u_\varepsilon), D^s u_\varepsilon)=0 \tag{2.5}$$

(also $(D^sD^3u_\varepsilon, D^su_\varepsilon)=0$). Now we apply lemma 1.1 and observe that

$$|(u_\varepsilon DD^su_\varepsilon, D^su_\varepsilon)|=\tfrac{1}{2}|(Du_\varepsilon D^su_\varepsilon, D^su_\varepsilon)|\leqq \tfrac{1}{2}\| Du_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbf{R}))}\| D^su_\varepsilon\|_0^2$$
$$\leqq c(\| u_0\|_2)\| D^s u_\varepsilon\|_0^2$$

(cf. [13]).

$$|(D^s(u_\varepsilon Du_\varepsilon), D^s u_\varepsilon)|\leqq 2c(\gamma,s)\| u_\varepsilon\|_s \| u_\varepsilon\|_{\gamma+1}\| D^s u_\varepsilon\|_0, \tag{2.6}$$

for some $\gamma>1/2$. We can take $\gamma=1$, and majorize $\| u_\varepsilon\|_s$ by $\| u_\varepsilon\|_0+\| D^su_\varepsilon\|_0$. Using (2.3) we obtain then:

$$\frac{1}{2}\frac{d}{dt}\| D^s u_\varepsilon\|_0^2\leqq c''\| D^s u_\varepsilon\|_0^2, \tag{2.7}$$

where c'' is some constant depending only on u_0, α, s and T. The result follows then from Gronwall's lemma. ■

2.2. *Other remarks.*

PROPOSITION 2.1 (A local existence-uniqueness result). *Let s be a real number $3/2<s<2$, and let $u_0\in H^s$. Then there exists $C_*=C_*(\alpha,s)$, such that*

‡ The scalar product in $L^2(\mathbf{R})$ (or resp. $L^2([0,1])$) is denoted (u,v).

the initial value problem (0.1), (0.2) *or* (0.2′), (0.3) *possesses a unique solution* u *weakly continuous from* $[0, T_*)$ *into* H^s, $T_* = (C_* \| u_0 \|_s)^{-1}$.

PROOF. We proceed as before using the regularized problem (2.1), (2.2). This time, since $s < 2$, instead of (2.3), we only infer from [13] that u_ε remains in a bounded set of H^1 as $\varepsilon \searrow 0$. It remains to show that u remains in a bounded set of $L^\infty(0, T_0; H^s)$, where $T_0 < T_*$ (C_* to be explicit).

We apply (2.6) with $s = \gamma + 1$ and instead of (2.7) we obtain some inequality

$$\frac{1}{2} \frac{d}{dt} \| D^s u_\varepsilon \|_0^2 \leqq C_* \| D^s u_\varepsilon \|_0^3$$

and by integration

$$\| D^s u_\varepsilon (t) \|_0 \leqq \frac{\| D^s u_{0\varepsilon} \|_0}{1 - C_* t \| D^s u_{0\varepsilon} \|_0}.$$

The result follows. ∎

REMARK 2.1. We do not know any global existence-uniqueness result in H $3/2 < s < 2$. Beside Theorem 2.1 and Proposition 2.1, the known results c existence of weak solutions are ([13]): existence of a global solution u ($L^\infty(0, T; H^1)$ if $u_0 \in H^1$; uniqueness of a solution $u \in L^\infty(0, T; H^s)$, $s > 3/2$. ∎

Under the assumptions of Theorem 2.1, let us now consider the mapping $\mathcal{A}_t$

$$u(0) = u_0 \mapsto u(t) \tag{2.8}$$

which is well defined from H^s into itself ($s \geqq 2$). This mapping is easily seen to b continuous and furthermore:

PROPOSITION 2.2. *$\mathcal{A}_t$ is locally Hölder continuous with exponent* 1/2, *fro* $H^{s+1/2}$ *into* $H^s (s \geqq 2)$. *More precisely, there exists a continuous function* φ *fro* $\mathbf{R}_+ \times \mathbf{R}_+$ *into* $\mathbf{R}_+$, *such that*

$$\| u(t) - v(t) \|_s \leqq \varphi \left(\| u_0 \|_{s+\frac{1}{2}}, \| v_0 \|_{s+\frac{1}{2}} \| u_0 - v_0 \|_{s+1/2}^{1/2} \right). \tag{2.9}$$

PROOF. It suffices to prove (2.9) for $u_{0\varepsilon}$, $v_{0\varepsilon}$, $u_\varepsilon(t)$, $v_\varepsilon(t)$, with a functic independent of ε (φ may depend on t). To do this, we set $w_{0\varepsilon} = u_{0\varepsilon} - v$ $w_\varepsilon = u_\varepsilon - v_\varepsilon$; w_ε satisfies the equation:

$$\frac{\partial w_\varepsilon}{\partial t} + \alpha \frac{\partial^3 w_\varepsilon}{\partial x^3} + \varepsilon \frac{\partial^4 w_\varepsilon}{\partial x^4} = - u_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - w_\varepsilon \frac{\partial v_\varepsilon}{\partial x}. \tag{2.10}$$

Applying D^s to each member of (2.10) and taking the L^2-scalar product wi $D^s w_\varepsilon$, we get

$$1/2\frac{d}{dt}\| D^s w_\varepsilon \|_0^2 + \varepsilon \| D^{s+2} w_\varepsilon \|_0^2 = -(D^s(u_\varepsilon D w_\varepsilon), D^s w_\varepsilon)$$

(2.11)

$$-(D^s(w_\varepsilon D v_\varepsilon), D^s w_\varepsilon).$$

Due to (1.7) the first term in the right-hand side of (2.11) is equal to $-(u_\varepsilon D^{s+1} w_\varepsilon, D^s w_\varepsilon)$, plus a remainder which is bounded by ((1.7) with $\gamma = 1$):

$$c(1,s)(\| u_\varepsilon \|_s \| D w_\varepsilon \|_1 + \| u_\varepsilon \|_2 \| D w_\varepsilon \|_{s-1}) \| D^s w_\varepsilon \|_0$$

$$\leqq c_1(s) \| u \|_s \| w_\varepsilon \|_s^2$$

By integration by parts, the term $-(u_\varepsilon D^{s+1} w_\varepsilon, D^s w_\varepsilon)$ is equal to (1/2) $(D u_\varepsilon D^s w_\varepsilon, D^s w_\varepsilon)$, and this is bounded by

$$c_2 \| u_\varepsilon \|_2 \| D^s w_\varepsilon \|_0^2.$$

Let us now majorize the second term in the right member of (2.11). Again, because of (1.7), this term is equal to $-(w_\varepsilon D^{s+1} v_\varepsilon, D^s w_\varepsilon)$ plus another expression bounded by

$$c(1,s)(\| w_\varepsilon \|_s \| v_\varepsilon \|_2 + \| w_\varepsilon \|_2 \| v_\varepsilon \|_s) \| D^s w_\varepsilon \|_0 \leqq c_3(s) \| v_\varepsilon \|_s \| w_\varepsilon \|_s^2.$$

Using Parseval's formula, we see that the term $(w_\varepsilon D^{s+1} v_\varepsilon, D^s w_\varepsilon)$ is equal to $(D^{s+1/2} v_\varepsilon, D^{1/2}(w_\varepsilon D^s w_\varepsilon))$ and this term is bounded in absolute value by

$$\| v_\varepsilon \|_{s+1/2} \| w_\varepsilon D^s w_\varepsilon \|_{1/2}.$$

By linear interpolation for the mapping $g \mapsto w_\varepsilon g, w_\varepsilon \in H^1$, we obtain the inequality:

$$\| w_\varepsilon g \|_{1/2} \leqq c_4 \| w_\varepsilon \|_1 \| g \|_{1/2}, \ \forall g \in H^{1/2};$$

$$\| w_\varepsilon D^s w_\varepsilon \|_{1/2} \leqq c_4 \| w_\varepsilon \|_1 \| D^s w_\varepsilon \|_{1/2}.$$

Finally

$$|(w_\varepsilon D^{s+1} v_\varepsilon, D^s w_\varepsilon)| \leqq c_4 \| v_\varepsilon \|_{s+1/2} (\| u_\varepsilon \|_{s+1/2} + \| v_\varepsilon \|_{s+1/2}) \| w_\varepsilon \|_1.$$

Now, a more precise form of (2.3) or (2.4) is that

$$\sup_{0 \leqq t \leqq T} \| u_\varepsilon(t) \|_2 \leqq \varphi_0(\| u_{0\varepsilon} \|_2) \tag{2.12}$$

$$\sup_{0 \leqq t \leqq T} \| u_\varepsilon(t) \|_s \leqq \varphi_1(\| u_{0\varepsilon} \|_s) \tag{2.13}$$

where $\varphi_0, \varphi_1, \cdots$, are continuous which may depend on T but not on ε (similar inequalities hold for v_ε). It is not difficult also to show that

$$\sup_{0 \leqq t \leqq T} \| w_\varepsilon(t) \|_1^2 \leqq \varphi_2(\| u_{0\varepsilon} \|_2, \| v_{0\varepsilon} \|_2) \| w_{0\varepsilon} \|_2^2 \tag{2.14}$$

(see for instance [9]).

Using all these majorations of the right-hand side of (2.11), we find now

$$\begin{aligned} \frac{d}{dt} \| D^s w_\varepsilon \|_0^2 &\leqq c_5(\| u_\varepsilon \|_s + \| v_\varepsilon \|_s) \| w_\varepsilon \|_s^2 \\ &\quad + c_6(\| u_\varepsilon \|_{s+1/2} + \| v_\varepsilon \|_{s+1/2})^2 \| w_\varepsilon \|_1 \\ &\leqq \varphi_3(\| u_{0\varepsilon} \|_{s+1/2} \| v_{0\varepsilon} \|_{s+1/2})(\| w_{0\varepsilon} \|_0 + \| w_\varepsilon \|_0^2 + \| D^s w_\varepsilon \|_0^2) \\ &\leqq \text{(with (2.14))} \\ &\leqq \varphi_4(\| u_{0\varepsilon} \|_{s+1/2} \| v_{0\varepsilon} \|_{s+1/2})(\| w_{0\varepsilon} \|_0 + \| w_{0\varepsilon} \|_0^2 + \| D^s w_\varepsilon \|_0^2). \end{aligned}$$

Using Gronwall's lemma, we conclude that

$$\| D^s w_\varepsilon(t) \|_0^2 \leqq \varphi_5(\| u_0 \|_{s+1/2}, \| v_0 \|_{s+1/2})(\| w_{0\varepsilon} \|_0 + \| w_{0\varepsilon} \|_0^2 + \| D^s w_{0\varepsilon} \|_0^2).$$

The proof is now completed.

REMARK 2.2.

i) We may, without any difficulty, consider a non-zero right-hand side in (0, 1), $f \in L^1(0, T; H^s)$. The results are valid as well for the backward problem $-T < t < 0$ with initial condition at $t = 0$: we have just to replace ε by $-\varepsilon$.

ii) It is easily checked with our results that the K. d. V. equation does not possess any regularizing nor any deregularizing effect in the spaces H^s:

(2.15) If $u_0 \in H^s$, $(s \geqq 2)$ and $u_0 \notin H^{s+\varepsilon}, \forall \varepsilon > 0$, then the same is true for $u(t)$, $\forall t > 0$;

(2.16) If $u_0 \in H^{s-\varepsilon}$, $\varepsilon > 0$ sufficiently small $(s > 2)$ and $u_0 \notin H^s$, then the same is true for $u(t)$, $\forall t > 0$.

For proving (2.15), observe that $u(t) \in H^s$, and if $u(t) \in H^{s+\varepsilon}$, $\varepsilon > 0$, then by solving the backward K. d. V. equation (see Remark 2.2 (i)), we should obtain $u_0 = u(0) \in H^{s+\varepsilon}$. The same proof for (2.16) holds.

iii) We note also that

(2.17) If $u_0 \in H^1$, $u_0 \notin H^2$, then the same is true for $u(t)$, $\forall t > 0$.

Indeed, if $u(t_0) \in H^2$ for some $t_0 > 0$, then by solving the backward problem we find a solution $\tilde{u} \in L^\infty(0, t_0; H^2)$ of the K.d.V. problem. The uniqueness theorem implies $\tilde{u} = u$, $\tilde{u}(0) = u_0 \in H^2$, in contradiction to the assumption.

iv) The same technique can be used for many other equations; generalized—in various ways—K. d. V. equations ([2], [10]), Euler equations (see [14]), first-order quasilinear hyperbolic equations. The technique applies also to the Sine–Gordon equation, but in this case the result similar to Theorem 2.1 was already obtained by J. C. Saut [9] using the non-linear interpolation [12].

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